4-POINT MOVING MEDIAN WITH CENTRING

Year, t	Sales, y ($1000)	Four-point moving median	Centred
1	7.3		
2	13.8		
		Median of (7.3, 13.8, 11.4, 11.2) = 11.3	
3	11.4		(11.3 + 12.6) ÷ 2 = 11.95
		Median of (13.8, 11.4, 11.2, 16) = 12.6	
4	11.2		(12.6 + 11.8) ÷ 2 = 12.2
		Median of (11.4, 11.2, 16, 12.2) = 11.8	
5	16		(11.8 + 11.7) ÷ 2 = 11.75
		Median of (11.2, 16, 12.2, 10.9) = 11.7	
6	12.2		(11.7 + 13.75) ÷ 2 = 12.725
		Median of (16, 12.2, 10.9, 15.3) = 13.75	

3-POINT MOVING AVERAGE

Year, t	Number of road deaths, y	Three-point moving average
1	225	
2	240	$\frac{225+240+201}{3}=222$
3	201	$\frac{240+201+192}{3}=211$
4	192	$\frac{201+192+185}{3}=192.67$
5	185	$\frac{192+185+160}{3}=179$
6	160	$\frac{185+160+172}{3}=172.33$
7	172	$\frac{160+172+127}{3}=153$
8	127	$\frac{172+127+132}{3}=143.67$
9	132	$\frac{127+132+101}{3}=120$
10	101	$\frac{132+101+100}{3}=111$
11	100	

STEPS TO DESEASONALISE DATA

1. Find the average for each year (yearly average)
2. Divide each month by the yearly average
3. Find the average of each month (seasonal index for each month)
4. Divide original data by its seasonal index

$$\text{Deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

Year	Sales figures quarter 1	Sales figures quarter 2	Sales figures quarter 3	Sales figures quarter 4
2010	5	7	9	3
2011	4	8	9	4
2012	5	9	10	5

1.

Year	Q1	Q2	Q3	Q4	Yearly average
2010	5	7	9	3	$\frac{5+7+9+3}{4}=6$
2011	4	8	9	4	$\frac{4+8+9+4}{4}=6.25$
2012	5	9	10	5	$\frac{5+9+10+5}{4}=7.25$

2.

Year	Q1	Q2	Q3	Q4
2010	$\frac{5}{6}=0.8333$	$\frac{7}{6}=1.1667$	$\frac{9}{6}=1.5000$	$\frac{3}{6}=0.5000$
2011	$\frac{4}{6.25}=0.6400$	$\frac{8}{6.25}=1.2800$	$\frac{9}{6.25}=1.4400$	$\frac{4}{6.25}=0.6400$
2012	$\frac{5}{7.25}=0.6897$	$\frac{9}{7.25}=1.2414$	$\frac{10}{7.25}=1.3793$	$\frac{5}{7.25}=0.6897$

3.

Year	Q1	Q2	Q3	Q4
2010	0.8333	1.1667	1.5000	0.5000
2011	0.6400	1.2800	1.4400	0.6400
2012	0.6897	1.2414	1.3793	0.6897
Total	2.1630	3.6881	4.3193	1.8297
SI	$\frac{2.1630}{3}=0.7210$	$\frac{3.6881}{3}=1.2294$	$\frac{4.3193}{3}=1.4398$	$\frac{1.8297}{3}=0.6099$

Year	Q1	Q2	Q3	Q4
Seasonal Index	0.7210	1.2294	1.4398	0.6099

4.

Year	Q1	Q2	Q3	Q4
2010	$\frac{5}{0.7210}=6.94$	$\frac{7}{1.2294}=5.69$	$\frac{9}{1.4398}=6.25$	$\frac{3}{0.6099}=4.92$
2011	$\frac{4}{0.7210}=5.55$	$\frac{8}{1.2294}=6.51$	$\frac{9}{1.4398}=6.25$	$\frac{4}{0.6099}=6.56$
2012	$\frac{5}{0.7210}=6.94$	$\frac{9}{1.2294}=7.32$	$\frac{10}{1.4398}=6.95$	$\frac{5}{0.6099}=8.20$

Year	Q1	Q2	Q3	Q4
2010	6.94	5.69	6.25	4.92
2011	5.55	6.51	6.25	6.56
2012	6.94	7.32	6.95	8.20

4-POINT MOVING AVERAGE - CENTRING

Year, t	Number of road deaths, y	Four-point moving average	4-moving average with centring
1	225		
2	240		
		$\frac{225+240+201+192}{4}=214.5$	
3	201		$\frac{214.5+204.5}{2}=209.5$ ←
		$\frac{240+201+192+185}{4}=204.5$	
4	192		$\frac{204.5+184.5}{2}=194.5$
		$\frac{201+192+185+160}{4}=184.5$	
5	185		$\frac{184.5+177.25}{2}=180.875$
		$\frac{192+185+160+172}{4}=177.25$	
6	160		$\frac{177.25+161}{2}=169.125$
		$\frac{185+160+172+127}{4}=161$	
7	172		$\frac{161+147.75}{2}=154.375$
		$\frac{160+172+127+132}{4}=147.75$	
8	127		$\frac{147.75+133}{2}=140.375$
		$\frac{172+127+132+101}{4}=133$	
9	132		$\frac{133+115}{2}=124$
		$\frac{127+132+101+100}{4}=115$	
10	101		
11	100		

TIME SERIES DATA

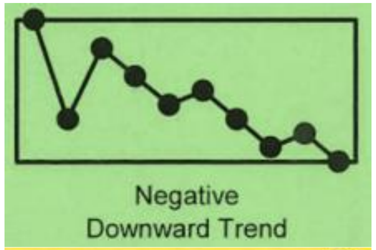
- Time on x-axis
- Other variable on y-axis (can be cost, sales, etc.)
- Can be described:
  - Trend
  - Seasonal Pattern
  - Cyclic/Irregular Pattern

RESPONSE VARIABLE

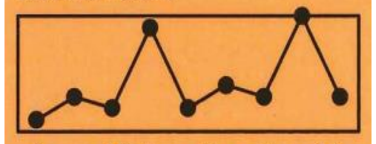
- Dependant
- Vertical (y-axis)
- Changes in response to the explanatory variable

EXPLANATORY VARIABLE

- Independent
- Horizontal (x-axis)

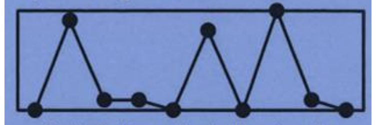


Seasonal Pattern



Seasonal Pattern with a pattern of 4 (i.e. 4PTMA is appropriate).

Cyclic / Irregular Pattern



Cyclic Pattern with significant peaks and troughs at irregular intervals.

Properties of Seasonal Indicies / Seasonal Index

Percentage Property: Converting seasonal indicies to percentages indicates performance above or below average for that season (e.g. 1.0505 → 105.05% → 5.05% above average for Period 2 and 0.7696 → 76.96% → 23.04% below average for Period 1).

Additive Property: The sum of the seasonal Indicies equals the number of seasons in the data (i.e. 0.7696 + 1.0505 + 1.1799 = 3).

Bouncing Ball Formulae

Below are shortcut formulae for the geometric sequence that models a ball dropped from an initial height a bouncing at r% efficiency.

Ball height after n^{th} bounce:

$Height = ar^n$

Total vertical distance travelled (S_∞):

$Distance = a\left(\frac{1+r}{1-r}\right)$

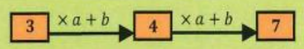
Vertical distance travelled up to n^{th} bounce:

$Distance = a\left(\frac{1+r-2r^n}{1-r}\right)$

r: bounce common ratio (as a decimal)
a: drop height
n: number of bounces

Recurrence Relation Example

A recurrence relation is defined as: $T_{n+1} = aT_n + b$ for some value of a and b. Find the recurrence relation of a sequence where the first three terms are 3, 4 and 7.



From diagram above, create two equations that links T_1 with T_2 and T_2 with T_3.

$T_2 = aT_1 + b \rightarrow 4 = 3a + b \ldots Equation\ 1$
$T_3 = aT_2 + b \rightarrow 7 = 4a + b \ldots Equation\ 2$

Using ClassPad, solve $Equation\ 1$ and $Equation\ 2$ to find a and b: $a = 3$ and $b = -5$
Substitute into $T_{n+1} = aT_n + b,\ T_1 = 3$
Hence $T_{n+1} = 3T_n - 5,\ T_1 = 3$

Arithmetic Sequence Examples

Some values of an arithmetic sequence are shown in the table below:

n	4	5	6	7
T_n	21.5	24.2	26.9	29.6

Find the explicit rule for the n^{th} term.

Need to determine a and d:
Calculating a: $a = 21.5 - (3 \times 2.7) = 13.4$
Calculating d: $d = 24.2 - 21.5 = 2.7$
Substitute values into $T_n = a + (n-1)d$
Hence, $T_n = 13.4 + (n-1) \times 2.7$

Find the recursive rule for the $(n+1)^{th}$ term.

From above, $a = 13.4$ and $r = 2.7$
Substitute values into $T_{n+1} = T_n + d,\ T_1 = a$
Hence, $T_{n+1} = T_n + 2.7,\ T_1 = 13.4$

Geometric Sequence Examples

Some values of a geometric sequence are shown in the table below:

n	3	4	5	6
T_n	0.5	2	8	32

Find the explicit rule for the n^{th} term.

$T_3 = ar^{3-1} = \frac{1}{2} \ldots Equation\ 1$
$T_4 = ar^{4-1} = 2 \ldots Equation\ 2$
Solve for a and r: $a = 0.03125$ and $r = 4$
Substitute into $T_n = ar^{n-1}$
Hence, $T_n = 0.03125 \times 4^{n-1}$

Find the recursive rule for the $(n+1)^{th}$ term.

From above, $a = 0.3125$ and $r = 4$
Substitute values into $T_{n+1} = T_n \times r,\ T_1 = a$
Hence, $T_{n+1} = 4T_n,\ T_1 = 0.03125$

Geometric Example

Seed sapling grows 80cm high in the first year. Growth rate halves each year. What will be the max height for the tree?
a = 80, r = ½ (0.5)

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{80}{1-0.5}$

= 160cm

Geometric Example

Stacey sold 50 items in her first year. Her sales increased by 20% each year. In which year did she sell over 2000 items?
a = 50, r = 120% (1.20)

$S_n = \frac{a(r^n-1)}{r-1}$

$2000 = \frac{50(1.2^n-1)}{1.2-1}$

Solve on calculator to find n
n = 12.5
= 13th year

Geometric Example

A population consists of 162 in the first year. If it decreases by 16% each year, how many will be left in the fifth year?
a = 162, r = 100% - 16% = 0.84
$t_n = ar^{n-1}$
$t_5 = 162 \times 0.84^{5-1}$
= 80.655
= 81

Arithmetic Example

First day, Jim receives 2 tokens on a game. The next day he receives 4, third day is 6, etc… If he plays the game every day in April, how many tokens will he get?
a = 2, d = 2, n= 30 (30 days in April)
$S_n = \frac{n}{2}(2a + (n-1)d)$
$S_{30} = \frac{30}{2}(2 \times 2 + (30-1) \times 2)$
= 930 tokens

Growth or Decay Sequences Formulae

Type	Explicit	Recursive
Growth (+)	$P_t = a(1+r)^t$	$P_{t+1} = (1+r)P_t$, $P_1 = a$
Decay (−)	$P_t = a(1-r)^t$	$P_{t+1} = (1-r)P_t$, $P_1 = a$

r: rate of growth or decay (as a decimal)
a: initial amount (i.e. P_1)
P_t: population at time t
t: time in years

Arithmetic and Geometric Sequences Formulae

Type	Explicit	Recursive	Sum of Series	Sum to Infinity
Arithmetic (+ or −)	$T_n = a + (n-1)d$	$T_{n+1} = T_n + d$, $T_1 = a$	$S_n = \frac{n}{2}(2a + (n-1)d)$	$S_\infty = \infty$ or $-\infty$
Geometric (× or ÷)	$T_n = ar^{n-1}$	$T_{n+1} = T_n \times r$, $T_1 = a$	$S_n = \frac{a(1-r^n)}{1-r}$	$S_\infty = \frac{a}{1-r}$

T_n: n^{th} term in the sequence
a: first term in the sequence (i.e. T_1)
r: common ratio between terms
d: common difference between terms
S_n: sum of the first n terms in the sequence
S_∞: sum of all possible terms in the sequence

Arithmetic Example

Bill is attempting to collect 300 footy cards. First month he gets 50 and every other month he gets 20 new cards.

a) How many cards will he get in 6 months?
a = 50, d = 20, n= 6
$t_n = a + (n-1)d$
$t_6 = 50 + (6-1) \times 20$
= 150 cards

b) How long will it take for him to collect all 300 cards?
$t_n = a + (n-1)d$
300 = 50 + (n-1) x 20
Solve on calculator to find n
n = 13.5 = 14 months

Correlation does not comply CAUSATION

If 2 variables have a strong correlation between them, it does not necessarily mean that one variable causes the other in reality (i.e. if the variables ice cream sales and number of deaths due to drowning have a strong correlation coefficient of 0.9, it doesn't mean that the 2 variables have a strong observable relationship in real life).

Causes of incorrect calculations of Pearson's correlation coefficient

- Coincidence: It could be a coincidence that data collected has a strong correlation. To reduce the chances of this, more data needs to be collected.
- Confounding: A third variable that was failed to be taken into account had a n influence between the 2 variables being tested

It also works in reverse, just because two variables have a weak correlation, due to coincidence and confounding, the 2 variables may in fact have a strong observable relationship in reality.

VALUE OF R	WHAT IT MEANS
1	Perfect positive
0.75 ≤ r < 1	Strong positive
0.5 ≤ r < 0.75	Moderate positive
0.25 ≤ r < 0.5	Weak positive
-0.25 ≤ r < 0.25	No relationship
-0.5 ≤ r < -0.25	Weak negative
-0.75 ≤ r < -0.5	Moderate negative
-1 ≤ r < -0.75	Strong negative
-1	Perfect negative

	Explanatory Variable		
Response Variable	Characteristic 1	Characteristic 2	
Characteristic 1			Row Sum
Characteristic 2			Row Sum
Characteristic 3			Row Sum
	Column Sum	Column Sum	

Pearson's Correlation Coefficient (r)

The value r such that $-1 \le r \le 1$ measures the direction and strength of a linear relationship between two variables.

Coefficient of Determination (r^2)

The value r^2 such that $0 \le r^2 \le 1$ shows the percentage of the variation in the response variable with the variation in the explanatory variable. It shows what percent of the data that is the closest to the line of best fit (i.e. if $r^2 = 0.85$, then 85% of the data is close to the line of best fit). Also, r^2 is equal to Pearson's Correlation Coefficient squared.

Least-Squares Line/Line of Best Fit ($y = ax + b$)

A linear equation that summarises the relationship between two variables where a is the gradient of the line (calculated by $a = rise/run$) and b is the y-intercept.

Long Term Steady State Solution

Two methods to find steady state solution:

- Substitute T_{n+1} and T_n with T and solve for T.
- Using ClassPad Sequences App, find a term for a large value of n (e.g. T_{50}) and look for a consistency.

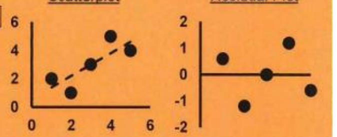
Calculate long term steady state solution for the sequence:
$t_{n+1} = 0.8t_n + 24, t_1 = 196$

- Add lists and spreadsheets page
- Generate sequence (menu, 3, 1)
- Type in sequence, enter
- Look at the values until the number starts to go steady = about 120

Residuals

Residual formula: $e = y - \hat{y}$

e: is the residual
y: is the observed value (y co-ordinate from the data)
$\hat{y}$: is the predicted value (substitute x co-ordinate into line of best fit equation)

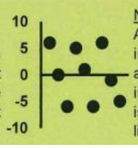


x	y	$\hat{y}$	e
1	2	1.4	0.6
2	1	2.2	-1.2
3	3	3	0
4	5	3.8	1.2
5	4	4.6	-0.6

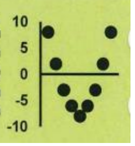
Step 1: Create a scatterplot and determine the correlation coefficient and the line of best fit (i.e. line of best fit is $y = 0.8x + 0.6$ and $r = 0.8$).
Step 2: determine the residual using the formula $e = y - \hat{y}$ and create a residual plot.
Step 3: analyse residual plot (i.e. random pattern indicates linear model is a good fit).

Residual Plots

Random Pattern: A random pattern in a residual plot indicates that the data is a good fit for a linear model.



Non-Random Pattern: A non-random pattern in a residual plot (such as a U-shaped pattern) indicates that the data is not a good fit for a linear model.



Two-Way Table

Displays data between two variables. Below is a two-way table showing the popularity of apples, bananas and peaches among males and females:

Fruit	Male	Female	Total
Apple	20	40	60
Banana	90	110	200
Peach	50	70	120
Total	160	220	380

What % of apples are liked by males?

$\frac{total\ likes\ of\ apples\ by\ males}{total\ apples} = \frac{20}{60} = 33.33\%$

What % of males or females don't like peaches?

$\frac{total\ likes\ of\ bananas\ and\ apples}{total\ males\ and\ females} = \frac{260}{380} = 68.42\%$

Construct a table of percentages:

Fruit	Male	Female	Total
Apple	5.26%	10.53%	15.79%
Banana	23.68%	28.95%	52.63%
Peach	13.16%	18.42%	31.58%
Total	42.11%	57.89%	100%

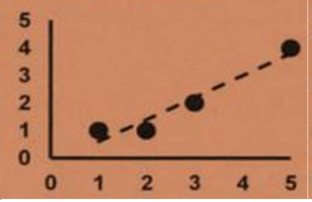
Extrapolation

Using the line of best fit to predict values that lie outside the range of the original data. Not recommended as the nature of the data beyond what was recorded is unknown (especially if the correlation coefficient is weak). Using the graph on the left, estimating the value of y when $x = 10$ is considered extrapolation as $x = 10$ lies outside the range of x values $(0-5)$.

$y = 0.8(10) - 0.2 = 7.8$
$\therefore (10, 7.8)$ is the extrapolated point.

Interpolation

Using the line of best fit to predict values that lie within the range of the data.



Line of best fit: $y = 0.8x - 0.2$

Estimating y when $x = 4$ can be determined by substituting $x = 4$ the line of best fit. This is considered interpolation as $x = 4$ is within the range of x values $(0-5)$.

$y = 0.8(4) - 0.2 = 3$
$\therefore (4, 3)$ is the interpolated point.

Maximum Flow
- Smaller of inflow and outflow capacity

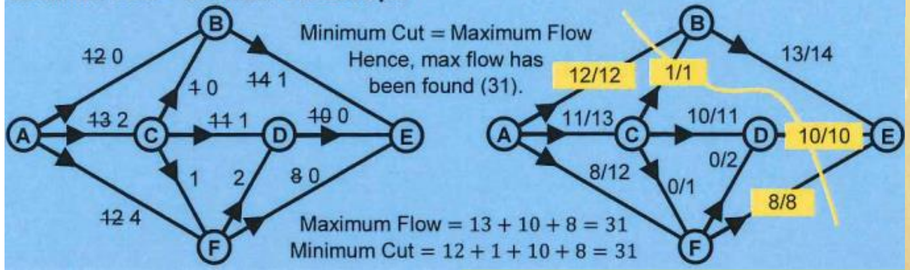
1) Identify cuts through the network
2) Find the capacity of each cut
3) Maximum flow = capacity of the minimum cut

Maximum Flow - Minimum Cut Theorem
- A cut is a line drawn through a number of edges which stops all flow from start to finish.
- Value of a cut is the total flow of the edges cut.
- If there is an edge flows toward the start of the network (source) rather than toward the finish (sink), it can be included in a cut and its flow can be treated as 0.

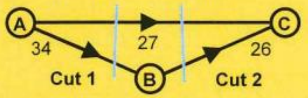
To determine the minimum cut, find the max flow through the network and create a cut that is as close to the start (source) as possible that passes through all edges with full flow.

Maximum Flow = Minimum Cut

Maximum Flow – Minimum Cut Example

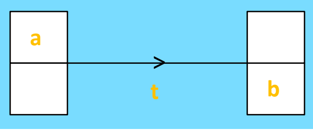


Step 1: Determine the value of all possible cuts.
Step 2: Find the value of the smallest cut; this is the maximum flow through the network.



Cut 1 = 34 + 27 = 61 and Cut 2 = 27 + 26 = 53.
Cut 2 is the smaller cut, hence 53 is maximum flow.

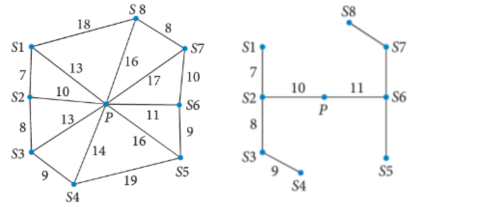
Float Time
- Max time an activity can be delayed without affecting the minimum completion project time (critical path)
- Float time = b – a – t



PRIMS ALGORITHM
1. Choose any vertex in the network
2. Find the lowest edge connecting this edge to another vertex
3. Look at all edges connecting these 2 vertices. Pick the edge of the lowest value
4. Look at all vertices covered and pick the edge with the lowest value
5. Repeat step 4 until all vertices have been included

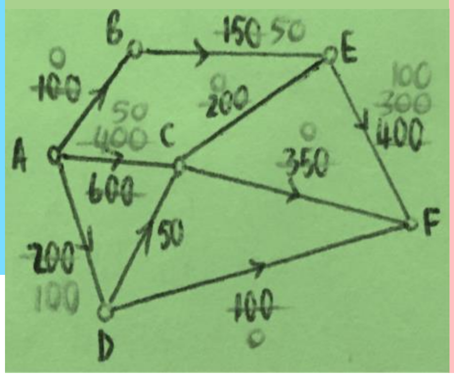
Example

Edgar connects electricity to 8 sites. Weighted graph below shows the direct distance between each site and the electricity source (P). Use prim's algorithm to find the <u>minimum</u> amount of electrical cable needed to power all 8 sites.

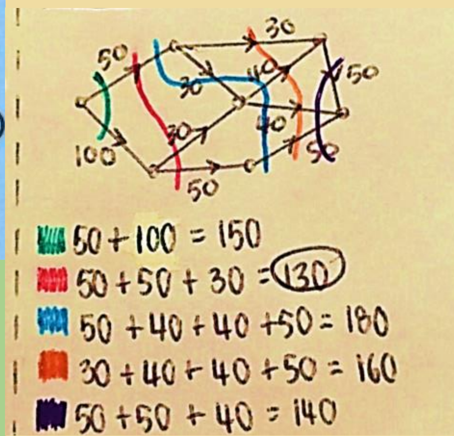


7 + 8 + 9 + 10 + 11 + 8 + 10 + 9 = 72 m

Systematic approach to find the maximum flow
ABEF = 100
ACEF = 200
ACF = 350
ADF = 100
Total max flow = 750L/minute



Maximum flow – minimum cuts



Maximum flow = 130L/min

<u>Trail</u>
- Can only travel along an edge once
- Open: Must travel along EVERY edge ONCE only
- Closed (cycle/circuit): Must travel along EVERY edge ONCE only and return to the starting vertex

<u>Spanning tree</u>
- Subgraph of another graph
- Connected graph, includes all vertices
- CANNOT contain loops, multiple edges or cycles/circuits

GRAPHS AND NETWORKS
In a network:
- Edges = Activities
- Vertices = Events

TERMINOLOGY

<u>Bridge</u>
- Edge that keeps the graph connected

<u>Connected graph</u>
- No isolated vertices
- Every vertex is **reachable** from every other vertex

<u>Complete graph</u>
- n vertices has $\frac{n(n-1)}{2}$ edges
- Every vertex connected to every other vertex

<u>Degree of a vertex</u>
- Loops counted twice
- Degree sum = 2 x the number of edges

<u>Adjacency matrix</u>
- Loops counted once

<u>Planar graph</u>
- No edges cross

<u>Euler's formula</u>
- V = E – F + 2
- E = V + F – 2
- F = E – V + 2

<u>Eulerian graph</u>
- ALL vertices must be of even degree
- Contains a CLOSED trail
- Repeated vertices, NO repeated edges
- Starts and finishes at the SAME vertex

<u>Semi-eulerian graph</u>
- ONLY 2 vertices can be of odd degree, the rest must be even
- Contains an OPEN trail
- Repeated vertices, NO repeated edges
- Starts and finishes at DIFFERENT vertices

<u>Directed graph/digraph</u>
- Has vertices and directed edges

<u>Walk</u>
- From each of the vertices, there is an edge to the next vertex in the sequence
- Can include repeated edges or vertices

<u>Path</u>
- All edges and vertices are different
- Open: Starts/finishes at DIFFERENT vertices
- Closed: Starts/finishes at the SAME vertex

<u>Hamiltonian path</u>
- Starts/finishes at DIFFERENT vertices
- Includes every vertex ONCE only

<u>Hamiltonian cycle</u>
- Starts/finishes at the SAME vertex

Includes every vertex ONCE only

<u>Tree</u>
- Connected graph
- CANNOT contain loops, multiple edges or cycles/circuits

HUNGARIAN ALGORITHM
Taxi company has three taxis and three customers (distance in km)

$$\begin{bmatrix} 11 & 19 & 17 \\ 21 & 15 & 13 \\ 15 & 18 & 21 \end{bmatrix}$$

a) Use the Hungarian algorithm to find the optimum allocation

1. **Row reduction**

$$\begin{bmatrix} 0 & 8 & 6 \\ 8 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}$$

2. **Column reduction**

$$\begin{bmatrix} 0 & 6 & 6 \\ 8 & 0 & 0 \\ 0 & 1 & 6 \end{bmatrix}$$

3. **Add smallest uncovered number to all covered values** (values with two lines through it get 2x the value of the uncovered number

$$\begin{bmatrix} 1 & 6 & 6 \\ 10 & 1 & 1 \\ 1 & 1 & 6 \end{bmatrix}$$

4. **Subtract the smallest number from all elements in the matrix**

$$\begin{bmatrix} 0 & 5 & 5 \\ 9 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Optimum allocation = A-1, B-3, C-2

A — 1
B — 2
C — 3

b) Find the total distance travelled
= 11 + 13 + 18
= 42 km

To MAXIMISE
1) Subtract every number in the matrix from the <u>largest</u> number
2) Proceed to use the row/column reduction and Hungarian algorithm

To MINIMISE
1) Subtract each number from the <u>smallest</u> number in <u>each row</u>
2) Proceed to use the row/column reduction and the Hungarian algorithm

- **If the number of rows and columns don't match, add a dummy row/ column of zeroes so then its equal**. DON'T INCLUDE THE DUMMY ZEROES IN FINAL ANSWER!

Loans
Borrowing a sum of money that needs to be paid back in full.

PV	Positive Value
PMT	Negative Value
FV	0

Investments
Investments are a deposit that grows over time due to interest, making regular contributions.

PV	Negative Value
PMT	Negative Value
FV	Positive Value

Annuities
Investment that pays all of it out over time through regular intervals.

PV	Negative Value
PMT	Positive Value
FV	0

Compound Interest Table Form
Investment: Lucas invests $1,000 into an account that pays 12% p.a. compounding monthly and makes monthly deposits of $200.

Month (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Deposit ($+r$)	Amount @ End (A_{n+1})
1	$1,000	+ $10	+ $200	$1,210.00
2	$1,210.00	+ $12.10	+ $200	$1,422.10
3	$1,422.10	+ $14.22	+ $200	$1,636.32

Loan: Sophia borrows $25,000 at 4% p.a. compounding weekly and makes weekly payments of $3,000 to pay off the loan.

Week (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Payment ($-r$)	Amount @ End (A_{n+1})
1	$25,000	+ $19.23	- $3,000	$22,019.23
2	$22,019.23	+ $16.94	- $3,000	$19,036.17
3	$19,036.17	+ $14.64	- $3,000	$16,050.81

Annuity: Charlotte invests $1,000 to buy an annuity that pays $200 per year at 7% p.a. compounding annually.

Year (n)	Amount @ Start (A_n)	Interest ($A_n \times \frac{i}{r}$)	Withdraw ($-r$)	Amount @ End (A_{n+1})
1	$1,000	+ $70	- $200	$870.00
2	$870.00	+ $60.90	- $200	$730.90
3	$730.90	+ $51.16	- $200	$582.06

Reducing Balance Loans
$A_n = \left(1 + \frac{r}{100}\right) A_{n-1} - monthly\ repayment$

Example

Bob borrows $140 000, interest rate at 7.5% p.a., compounded, adjusted monthly

a) Find the monthly repayment if it is to be repaid in 15 years

N = 12 x 15 = 180
I = 7.5
PV = 140 000
Pmt = 1297.82
FV = 0
Ppy/Cpy = 12

b) Find the total amount repaid

= 180 x 1297.82
= $233 608

c) What was the total interest paid?

= amount paid – amount borrowed
= 233 608 – 140 000
= $93 608

Annuity Investment Example
Kim invests $3000 at 6.5% p.a. interest, compounded monthly. At the end of each month, she makes a deposit of $800. Find the amount of her investment after 5 years.

N = 5 x 12 = 60
I = 6.5
PV = - 3000
Pmt = - 800
FV = 60 687.63
Ppy/Cpy = 12
The investment value after 5 years is $60 687.63

Changing terms of a loan Example
Borrows $100 000, **8%** compounded monthly, to be repaid over 10 years, repayments of $1213.28... After 5 years, interest rate reduces to **7.5%.**

a) Find the new repayment to fully pay off the loan in 10 years

N = 5 x 12 = 60
I = **8**
PV = 100 000
Pmt = - 1213.28
FV = - 59 836.57
Ppy/Cpy = 12

N = 5 x 12 = 60
I = **7.5**
PV = 59 836.57
Pmt = - 1199
FV = 0
Ppy/Cpy = 12

b) How much is saved due to the interest rate cut?

Total repaid x new repayment
= 60 x 1213.28
= $72 796.80
Total repaid x new repayment
= 60 x 1199
= $ 71 940
Interest saved
= 72 796.80 – 71 940
= $856.80

Perpetuities
- Regular payments are paid from the investment that continues forever

$Q = \frac{PR}{100}$

Q = Value of the regular payment per period
P = Amount invested
R = Percentage interest rate per period

Example

$500 000 perpetuity, 3.5% p.a, compounded annually. How much will they receive each year?

$Q = \frac{500000 \times 3.5}{100}$ = $17 500

Annuity Example
$400 000 annuity, interest of 8% p.a., compounded monthly, making monthly repayments of $3500. How long will the annuity last?

N = 215.98 = 216 months
I = 8
PV = - 400 000
Pmt = 3500
FV = 0
Ppy/Cpy = 12

Simple Interest
- Interest calculated as a % of the principal

$I = \frac{PRN}{100}$

A = I + P (where A is the final amount/value of an investment

Compound Interest
- Exponential increasing pattern (interest increases over time)

Simple Interest
- Linear pattern (interest remains constant over time)

Compound Interest
- Interest added to the principal and then the total amount is reinvested

Yearly compound interest

$A = P\left(1 + \frac{r}{100}\right)^n$

Adjusted compound interest

$A = P\left(1 + \frac{r}{n}\right)^n$

I = A - P (where A is the final amount/value of an investment

Effective Interest Rate (EIR)
$e = \left(1 + \frac{i}{n}\right)^n - 1$

e = effective yearly interest rate
i = nominal yearly interest rate
n = number of compounding periods in a year

Example

What's the EIR if the nominal interest rate is 7% p.a, compounded monthly?

$e = \left(1 + \frac{0.07}{12}\right)^{12} - 1$
= 0.07229
= 7.23% p.a.

Frequency of Compounding Interest
The more times interest compounds per year, the more interest is earned. The higher the value of n, the higher the effective annual rate of interest. There is diminishing returns on interest gained as n increases.

n	i	Converting to effective annual rate → $i_{effective}$
Yearly (1)	5%	5%
Half-Yearly (2)	5%	5.062%
Quarterly (4)	5%	5.095%
Monthly (12)	5%	5.116%
Fortnightly (26)	5%	5.122%
Weekly (52)	5%	5.125%
Daily (365)	5%	5.127%

ClassPad Compound Interest Examples
Jackson borrows $20,000 at 12% p.a. compounding monthly. He pays $350 every month to pay off the loan. How much would he still owe after 5 years of making payments?

N	60
I%	12
PV	20000
PMT	-350
FV	-7749.55
P/Y	12
C/Y	12

Emily borrows $25,000 at a rate of 12% p.a. compounding half-yearly. Her loan needs to be repaid in 4 years. What are Emily's half-yearly repayments?

N	8
I%	12
PV	25000
PMT	-4025.90
FV	0
P/Y	4
C/Y	4

James borrows $50,000 and is to be fully repaid in monthly repayments of $485.60 for 12 years. If interest is compounded monthly, determine the annual rate of interest.

N	144
I%	5.91%
PV	50000
PMT	-485.60
FV	0
P/Y	12
C/Y	12

Lily invests $10,000 at 7% p.a. compounding half-yearly. Lily wants her account to reach $50,000 in 10 years. How much does she need to deposit every six months?

N	20
I%	7
PV	-10000
PMT	-1064.44
FV	50000
P/Y	2
C/Y	2

Lachlan invests $2,000 and adds $200 to his account every quarter. Interest rate is 3.2% p.a. compounding quarterly. Determine how much is in his account in 5 years.

N	20
I%	3.2
PV	-2000
PMT	-200
FV	6664.63
P/Y	4
C/Y	4

Grace invests $700,000 to buy an annuity that pays $50,000 at 5.4% p.a. compounding annually. How many years will Grace be able to withdraw money?

N	26.82
I%	5.4
PV	-700000
PMT	50000
FV	0
P/Y	1
C/Y	1